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ABSTRACT

This paper illustrates a different conception of arithmetic, an arithmetic with reasons as well as rules. This arithmetic includes making connections between different representations and making sense of rules as well as using them. It provides a foundation for "Algebra the Web of Knowledge and Skill" in the sense that algebra can be seen as the generalization, formalization, and symbolization of this arithmetic. Past and recent views of arithmetic are contrasted in order to make the differences between them explicit. Arithmetic yet to come is also considered. (Contains 27 references.) (ASK)

An arithmetic carol

Cathy Kessel

Ours is not to reason why, just invert and multiply.

Minus times minus equals a plus, the reasons for this we need not discuss.

The lines above are mnemonics for rules of arithmetic: (1) when dividing by a fraction, invert and multiply; and (2) the product of two negative numbers is positive. They suggest a sinister subtext: arithmetic consists of rules without reasons. These mnemonics for remembering how to perform arithmetic operations without understanding capture the knowledge that is the aim of what Kaput (1999) might call "Arithmetic the Institution"—mechanical procedures without a substantive grounding in meaning or understanding. A consequence of experiencing this arithmetic is that when students encounter algebra, they find themselves completely unprepared for it: algebra is not a generalization and symbolization of the arithmetic they know, it is an alien if not alienating experience.

I would like to illustrate a very different conception of arithmetic, what Kaput might call Arithmetic the Web of Knowledge and Skill—an arithmetic with reasons as well as rules. This arithmetic includes making connections among different representations and making sense of rules as well as using them. It provides a foundation for Algebra the Web of Knowledge and Skill in the sense that algebra can be seen as the generalization, formalization, and symbolization of this arithmetic.

This is not the arithmetic that most teachers (and parents and school administrators) learned. It is not the arithmetic that many experienced teachers are used to teaching. For all of us, new views of arithmetic are filtered through past views of arithmetic—not *long* past, but *our* past. By contrasting past and recent views of arithmetic, I attempt to make the differences between them explicit.

Arithmetic Past

Arithmetic the Institution is consistent with the idea that arithmetic algorithms are rules without reasons. Ginsburg and Yamamoto wrote in

1986, "It is no exaggeration to say that the vast majority of elementary school students are forced to spend most of their instructional time on two topics: the number combinations and the standard calculational algorithms. For them, arithmetic *is* these topics and these only." In this view, to know arithmetic is to know number facts and be able to perform computational procedures correctly.[1]

Subtraction. Before they enter school, children begin learning to count and recognize numbers, and most can solve very simple word problems (see Fuson, 1988, p. 293 for details). In school, children develop a wide variety of strategies for solving addition and subtraction problems but these strategies are not supported by instruction, instead textbooks present very limited concepts of addition and subtraction. Textbook word problems focus on the interpretation of subtraction as "take-away" and the associated solution procedure of "sum – addend = [answer]" (Fuson, 1992). The strategy a child uses for a subtraction problem seems to be associated with its meaning—children who view subtraction as "take-away" tend to use a counting down strategy rather than the less error-inducing counting up (Fuson, 1992). A counting down solution for 15 – 8 = ? might be to take 8 from 15 by counting 8 numbers down from 15—"14, 13, . . . , 8, 7."

A different meaning that might be attached to 15 – 8 = ? is "How much *more* is 15 than 8?" This suggests starting with 8 and finding out how much more is needed, which is what children do when counting up. This meaning is not a focus of first and second grade problems in U.S. textbooks—these focus instead on the "take-away" interpretation. Children experience other subtraction situations and associated solution procedures later or considerably less often (Fuson, 1988).

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[1]Kamii explains why viewing a statement like "4 + 2 = 6" as a number fact is counter to the theory developed from the research of Piaget and his co-workers. She views such statements as relationships between constructed quantities rather than as observable facts. (Kamii adds that 4 + 2 can be known by counting, but that counting and knowing by deduction are different.) She illustrates the difference between an observable fact and a relationship with the results of the class-inclusion tasks developed by Piaget. In the class-inclusion task, an interviewer gives a child some objects, for example, six miniature dogs and two cats of the same size and asks a series of questions to insure that he or she understands the meaning of "*all* the animals," "*all* the dogs," and "*all* the cats." Then the interviewer asks "Are there more dogs or more animals?" "More dogs" is the answer of a typical four-year-old who responds to explanations that there are more animals with puzzled looks (Kamii & DeClark, 1985, pp. 12, 67). Kamii explains "Observation of facts is one thing, and the logico-mathematization of what is observable is quite another." (For further discussion of this issue see Kamii & DeClark, 1985, pp. 64–73.) Post-Piagetian research suggests that such responses are heavily dependent on language and context (see Donaldson, 1990).

First and second grade textbooks have many tasks like these (Fuson, 1988):

$$3 + 2 = ?$$
$$15 - 8 = ?$$

Both tasks like these and the interpretation of subtraction as "take-away" focus on the interpretation of the equal sign as "results in" rather than "the same as."

Later, students learn the multi-digit subtraction algorithm (subtraction with "borrowing"). After extensive analysis of student errors, VanLehn wrote, "Most elementary students have only a dim conception of the underlying semantics of subtraction, which are rooted in the base ten representation of number. When compared to the procedures they use to operate vending machines or play games, subtraction is as dry, formal, and disconnected from everyday interests as the nonsense syllables used in early psychological investigations were different from real words" (1983, p. 201).

Algebra. As they learn arithmetic, students find that when they add or multiply two numbers, the result is the same no matter which number is first. Symbolized algebraically, these observations are the commutative laws for addition and multiplication:[2]

$$a + b = b + a \text{ and } ab = ba.$$

Although beginning algebra students may have made the generalization that "the order in which you add or multiply two numbers doesn't matter," they often see these algebraic formalizations as alien. In these equations the equal sign expresses equivalence. The textbook examples above show that arithmetic syntax for the equal sign is quite different—it is a "do-something signal" and "the right side of the equal sign should indicate the answer" (Kieran, 1990, p. 100). The equations expressing the commutative laws don't have "an answer" on the right side of the equal sign, instead the equations themselves are objects expressing information about numbers. To a beginning algebra student, the equal sign says "Do something," but there is nothing to do.

[2]More formally, $\forall a \forall b(a + b = b + a)$ and $\forall a \forall b(ab = ba)$. One might be even more formal and specify the range of the quantifiers.

Viewing the equal sign as a "do-something" signal persists into college (Kieran, 1990). Students sometimes put "=" in situations where mathematicians would write "=>" (an implication sign). For instance, college students sometimes write:

$$g(x) = x^2 = g'(x) = 2x$$

instead of:

$$g(x) = x^2 \Rightarrow g'(x) = 2x.$$

This is an example of generalizing from arithmetic experiences—but though these generalizations make sense in terms of the arithmetic syntax derived from the meaning of the equal sign as "results in," they do not if the symbols have their standard mathematical meanings.

In this view, arithmetic is about number computations, and algebra is about solving equations with letters; arithmetic is about processes, and algebra is about objects. Arithmetic and algebra are two separate layers of a cake (Kaput, 1999, 1995; Steen, 1990)—and though there is something cookbook about both of them, they've been made from different recipes.

Geometry. Geometry is often the next layer of the cake. Lampert (1986, p. 327) points out that fourth graders have difficulty separating ideas about length and area adequately. Research on geometric representations suggests that students continue to have trouble with length and area. Clinical studies of ninth graders (Fuys, Geddes, & Tischler, 1988, p. 113) revealed two common misunderstandings "(1) thinking of area as related to angle sums and (2) confusing area with perimeter." The latter sometimes led students to misapply the formulas for rectangle area and perimeter that they had memorized. Kieran (1990) summarizes studies documenting North American middle school and high students' difficulties with representations that use length and area. These difficulties may well be related to students' difficulties with graphical representations of functions. Not surprisingly, college students seem "reluctant to visualize" (Eisenberg & Dreyfus, 1991). Eisenberg and Dreyfus (1991) say,

> The work of [Ferrini-]Mundy, Dick, Monk, Swan, and Vinner supports the preliminary finding by the authors that [calculus] students have a strong tendency to think

> algebraically rather than visually. Moreover, this is so even if they are explicitly and forcefully pushed towards visual processing. . . . What is needed to generate a visual understanding which would include the ability to solve problems utilizing both visual and analytic thinking in concordance with one another, and help students feel at ease in both domains? (p. 29)

Connections between algebra and geometry are fundamental in calculus. For example, one important idea is: approximating the area under a curve by a collection of rectangles, writing the area of those rectangles as an algebraic expression, and understanding how changes in the algebraic expression are related to geometric changes in the corresponding graphical representation. Variants of this idea appear in the Fundamental Theorem of Calculus, methods of approximating integrals, and methods of calculating volumes.

Compared with the abundance of research on children's counting and computation, little effort seems to have been devoted to children's geometrical thinking and its relationship with instruction (Hershkowitz, 1990, p. 93). The van Hiele Model of Thinking in Geometry Among Adolescents Project (Fuys, Geddes, & Tischler, 1988) investigated three K–8 textbook series and found that most textbook exercises related to geometry involved activities in which "the student identifies, names, compares and operates on geometric figures . . . according to their appearance" rather than analyzing figures, logically interrelating their properties, or other more advanced activities. Visualization and geometry are not considered important parts of mathematics, particularly elementary mathematics.

This brief sketch of precollege mathematics as it is often conceived illustrates some of the discontinuities between Arithmetic the Institution and Algebra the Institution, between students' preschool and school experiences, and between students' precalculus and calculus experiences, namely that:

- Children enter school knowing how to solve simple word problems, but
  early arithmetic does not build on what students know.

- Early subtraction focuses on the "take-away" interpretation, and
  subtraction with borrowing has little semantic meaning.

- In arithmetic, the equal sign means "do something," but in algebra, the equal sign means "the same as."

- Geometry is not connected with arithmetic or algebra, but connections between geometry and algebra are an important aspect of calculus.

Arithmetic Present

For most adults educated in the United States, the arithmetic I've sketched is Arithmetic Past. For most people not directly involved with mathematics education, this arithmetic is not the ghost of a departed quantity—it is very much Arithmetic Present.

For many involved with mathematics education, arithmetic is in a state of transition. The National Council of Teachers of Mathematics has drawn on research such as that I've described in making its 1989 *Curriculum and Evaluation Standards for School Mathematics*. However, the U.S. curriculum as reflected in recent textbooks is "a splintered vision" (Schmidt, McKnight, & Raizen, 1997).

I now turn to the arithmetic that for many of us is Yet To Come.

Arithmetic Yet To Come

Pupil: And I to youre authoritie my witte doe subdue,
whatsoever you say, I take for true.
Master: Thoughe I mighte of my Scholler some credence require,
yet except I shew reason, I do not it desire.

Robert Recorde, *The Grounde of Artes* (arithmetic book published in 1540)
(quoted in Cajori, 1957, p. 184)

I'd like to contrast the arithmetic of number facts and computational procedures with sketches of a very different arithmetic in which students:

- explain their reasoning verbally and symbolically;
- make sense of different solutions;

- make sense of computational procedures;
- make connections between different representations.

The examples that follow are meant to be illustrations of what it might mean to know arithmetic rather than prescriptions for teaching. I do not discuss the (considerable) pedagogical knowledge and skill that are necessary to make these situations occur in classrooms.

Several of my examples are drawn from the research of Liping Ma who has been an elementary student, elementary teacher, and education researcher in China. She points out that "knowing arithmetic" has very different meanings in China and the United States. For example, in China the distributive law is first learned in terms of arithmetic—students learn numerical versions of $a(b + c) = ab + ac$. Associated with this different conception of arithmetic is a different relationship between arithmetic and algebra. As a student, Ma found arithmetic much harder than algebra and thinks this may be a common experience for Chinese students. In Chinese "algebra" means "replace numbers with letters." The characters for the word *algebra* are "number" and "replace" (Ma, personal communication, September 18, 1997). This suggests that in China, among many other differences, algebra is closer to being generalized arithmetic than it is in the United States. As documented by numerous cross-national studies, this curricular approach does not hinder development of computational skills (for a review see Cai, 1995). Chinese students out-perform U.S. students by a wide margin on tests of computation.

Verbal and symbolic reasoning in subtraction. Teacher Li (a pseudonym) described a particular stage in teaching her students subtraction:

> We start with the problems of a two-digit number minus a one-digit number, such as 34 – 6 =. I put the problem on the board and ask students to solve the problem on their own. . . . After a few minutes they finish. I have them report to the class what they did. They might report a variety of ways. One student might say, "34 – 6, 4 is not enough to subtract 6. But I can take off 4 first, get 30. Then I still need to take 2 off because 6 = 4 + 2. I subtract 2 from 30 and get 28. So, my way is
>
> $$34 - 6 = 34 - 4 - 2 = 30 - 2 = 28."$$

Teacher Li, like many experienced Chinese, Japanese, and Taiwanese elementary teachers, tends to focus on student responses to the problems she poses (Ma, 1999; Stevenson & Stigler, 1991; Stigler, Fernandez, & Yoshida, 1996), so it is not surprising that she was able to describe possible responses in detail. Here the hypothetical student has decomposed 6 as 4 + 2 and has correctly distributed –1, i.e. rewritten 34 – 6 as
34 – 4 – 2. An intermediate step might be:

$$34 - 4 - 2 = (34 - 4) - 2.$$

Thinking of $34 - 4 - 2$ as $34 + (-4) + (-2)$ suggests why this example could be part of the foundation for the algebraic associative law for addition:

$$a + b + c = (a + b) + c = a + (b + c).$$

Teacher Li's description continues:

> Another student who worked with sticks might say, "When I saw that I did not have enough separate sticks, I broke one bundle. I got 10 sticks and I put 6 of them away. There were 4 left. I put the 4 sticks with the original 4 sticks together and got 8. I still have another two bundles of tens. Putting the sticks left all together, I had 28." Some students might report. . . . "We have learned how to compute 14 – 8, 14 – 9, why don't we use that knowledge? So, in my mind I computed the problem in a simple way. I regrouped 34 into 20 and 14. Then I subtracted 6 from 14 and got 8. Of course I did not forget the 20, so I got 28." (Ma, 1999)

One way of writing the third method might be (Ma, personal communication, September, 18, 1997):

$$
\begin{aligned}
34 &= 20 + 14 \\
* \; 34 - 6 &= 20 + 14 - 6 \\
14 - 6 &= 8 \\
* \; 34 - 6 &= 20 + (14 - 6) \\
&= 20 + 8 \\
&= 28.
\end{aligned}
$$

Ma adds that teachers like Li ask their students, even first and second graders, to make such mathematical arguments.

This computation (and the decomposition of 6 into 4 + 2 in the first method) illustrates several ideas. One is that of rewriting an expression in an equivalent, but more convenient, form—something often done in algebra. In this computation, rewriting 34 as 20 + 14 is more convenient because the hypothetical student knows how to compute 14 – 6. This computation also uses substitution: 8 is substituted for (14 – 6). All three methods involve multi-step calculations in which the equal sign (written or implicit) acts more as an equivalence than a "do-something" signal and the right side of the equal sign does not always indicate the answer.

The third method might also be written like this:

$$34 - 6 = (20 + 14) - 6 = 20 + (14 - 6) = 20 + 8 = 28.$$

Generalizing the transformation involved in going from the second equality to the third yields the associative law for addition.

Making sense of multi-digit multiplication algorithms symbolically. In sixth grade Rebecca Corwin (1989) invented a checking method for the standard multiplication algorithm. Her method always gave correct results but was banned by her teacher.

Standard algorithm	Corwin's method
34	34
×23	×23
102	92
680	690
782	782

The first partial product of the standard algorithm is 3 x 34 = 102 (first digit of the multiplier times the multiplicand). The second partial product is 20 x 34 = 680 (second digit of the multiplier times the multiplicand).

In Corwin's method, the first partial product is 4 x 23 = 92 (first digit of the multiplicand times the multiplier). The second partial product is 30 x 23 = 690 (second digit of the multiplicand times the multiplier). These partial products occur when the standard algorithm is used—on the problem 23 x 34, not 34 x 23. Because multiplication is commutative, 23 x 34 is equal to 34 x 23, so Corwin's method gives the same result as the standard algorithm.

U.S. elementary teachers sometimes view the zero in the second partial product of the multiplication algorithm as a place-holder and sometimes as a distraction (Ma, 1999). Sometimes students are taught the standard algorithm without the zero:

$$\begin{array}{r} 34 \\ \underline{\times 23} \\ 102 \\ \underline{68\ \ } \\ 782 \end{array}$$

Teacher Chen (a pseudonym), an experienced Chinese elementary teacher, suggested helping students to understand the multiplication algorithm by guiding them to find all the possible ways in which the columns may be lined up (Ma, 1999). Applying his idea to the previous example would lead students to consider the following:

34	34	34	34
×23	×23	×23	×23
69	92	102	68
92	69	68	102
782	782	782	782

Lampert (1986, pp. 330–337) describes a symbolic representation she used in her fourth grade class to make sense of the standard multiplication algorithm. In this representation the calculation is:

```
  34 ————————> 30 + 4
x 23
 690 <———————— 30 x 23
+ 92 <———————— 4 x 23
 782
```

As Lampert's representation suggests, the equivalence of Corwin's method and the standard algorithm (with or without the zero) can be explained using the distributive law. The formulation of the distributive law used in the United States is:

$$a(b + c) = ab + ac.$$

Decomposing 23 as 20 + 3 and distributing the 34 yields:

$$(34)(23) = (34)(20 + 3) = (34)(20) + (34)(3).$$

Chinese elementary students would recognize the first equality as justified by the distributive law (Ma, 1999). This form shows the partial products 34 x 20 = 680 and 34 x 3 = 102 that are obtained from the standard algorithm—and shows the reason for the zero.

Decomposing 34 as 30 + 4 gives:

$$(30)(23) + (4)(23) = (30)(20 + 3) + (4)(20 + 3).$$

Distributing the 30 and the 4 yields:

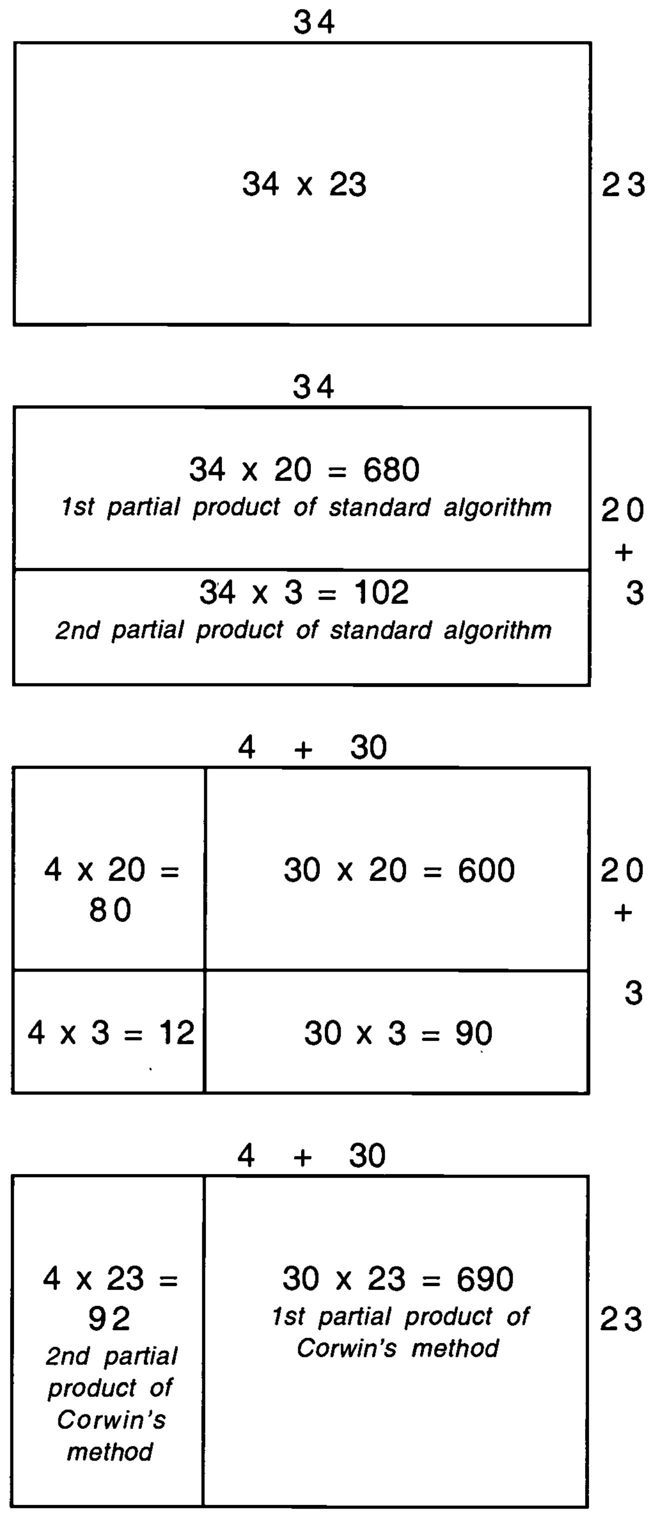
$$(30)(20) + (30)(3) + (4)(20) + (4)(3).$$

Collecting terms (using the distributive law):

$$(30)(20 + 3) + (4)(20 + 3) = (30)(23) + (4)(23).$$

This shows the partial products 30 x 23 = 690 and 4 x 23 = 92 of Corwin's method.

Making sense of the multiplication algorithm visually: Connecting symbolic and visual representations. Lampert uses drawings of objects in groups (e.g., butterflies in jars or astronauts on planets) to help students make sense visually of the distributive law and multiplication. Students familar with area and perimeter could also use the following geometric decompositions corresponding to the equations above. (Note: Graphics are not to scale.)



Making sense of negative number multiplication symbolically. In 1797, the fourteen-year-old Marie-Henri Beyle (who later became the novelist

Stendhal) loved mathematics, but had trouble understanding multiplication with negative numbers.

> In my view, hypocrisy was impossible in mathematics and, in my youthful simplicity, I thought it must be so in all the sciences to which, as I had been told, they were applied. What a shock for me to discover that nobody could explain to me how it happened that: minus multiplied by minus equals plus
> (– x – = +)! (Stendhal, 1958, pp. 257–258)

Multiplication with negative numbers can be explained using the distributive law. Before tackling Stendhal's problem of the product of two negative numbers, it helps to see what the product of a positive and a negative number should be. (As Klein (1908/1945, p. 27) points out, in this use of the distributive law we are extending rules for whole numbers to the case of negative numbers.) Take any numbers, say, 2 and 3.

$3 + -3 = 0$, so 2 times $(3 + -3)$ is also 0, i.e.,

$$(2)(3 + -3) = (2)(0) = 0.$$

Using the distributive law on the left side of the equation above:

$$(2)(3 + -3) = (2)(3) + (2)(-3) = 0$$

$$6 + (2)(-3) = 0.$$

So $(2)(-3)$ must be -6, i.e. $(2)(-3) = -(2)(3)$.

The same idea works for seeing what the product of two negative numbers should be.

$3 + -3 = 0$, so -2 times $(3 + -3)$ is also 0, i.e.,

$$(-2)(3 + -3) = (-2)(0) = 0.$$

Using the distributive law on the left side of this equation yields

$$(-2)(3 + -3) = (-2)(3) + (-2)(-3) = 0$$

$$-6 + (-2)(-3) = 0.$$

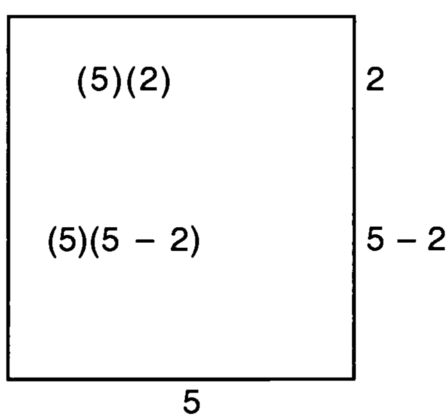
So (–2)(–3) must be 6.

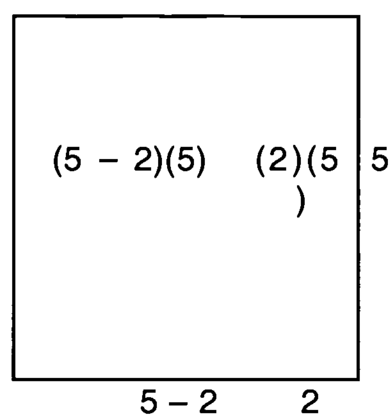
This justification can only be used if students are familar with the arithmetic version of the distributive law. By not establishing a basis for explanations like these, Arithmetic Past makes it difficult if not impossible for students to be given a reason why the product of two negative numbers is positive.

Making sense of negative number multiplication visually. Negative number multiplication can also be given a geometric explanation. Consider (5 – 2)(5 – 2).

$$(5 - 2)(5 - 2) = (3)(3) = 9.$$

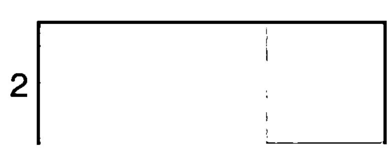
In terms of area, this might be visualized in terms of a square with sides of length 5, hence area 25. The square can be dissected in various ways, among them:

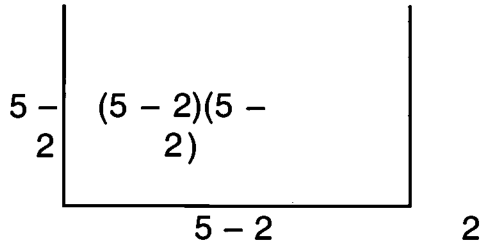




Subtracting the shaded areas yields the lower left area (5 – 2)(5 – 2) except that the overlap has been subtracted twice, so the area of the upper right square, (2)(2), should be added to the result, i.e.,

$$(5 - 2)(5 - 2) = (5)(5) - (2)(5) - (2)(5) + (2)(2).$$





Using the distributive law to expand (5 – 2)(5 – 2) yields:

$$(5 - 2)(5 - 2) = (5)(5) - (2)(5) - (2)(5) + (-2)(-2).$$

Comparing the two expressions yields:

$$(2)(2) = (-2)(-2).$$

Concluding discussion

Arithmetic the Institution and Algebra the Institution have two major discontinuities. The first occurs when children enter school: Addition and subtraction tend to be taught in a manner that ignores students' strategies and focuses on a limited interpretation of subtraction. The second discontinuity is the change in the interpretation of the equal sign from arithmetic to algebra. Moreover, visual representations involving area and perimeter are a peripheral part of Arithmetic Past. Unless students learn about this topic on their own, they are left to struggle when they encounter sophisticated manifestations of area and perimeter in graphing and calculus. Perhaps most damaging of all, Arithmetic Past is a collection of rules without reasons. Making sense of those rules is not a part of what it means to know Arithmetic Past.

By denying them access to important mathematical ideas, Arithmetic Past has bound many students in chains of ignorance, and sometimes fear, to which algebra and calculus often add links. Many of those students have become adults who, understandably, have not reflected on their arithmetic past. Their reactions to reform sometimes suggest that they are haunted by the Ghost of Arithmetic Past in their interpretations of Arithmetic Yet To Come and in their expectations for children's learning. The Ghost of Arithmetic Past also casts its shadow over teachers, dimming their expectations of students.

> Are these the shadows of things that Will be, or are they the shadows of what May be, only? . . . courses will

> foreshadow certain ends, to which if persevered in, they must lead. But if the courses be departed from, the ends will change. (Dickens, 1843/1956, pp. 149-150)

Arithmetic Past need not be the shadow of what will be. Instead:

- arithmetic can be reconceived to include reasoning, sense-making, and connecting representations (both symbolic and visual);
- algebra can be conceived as a generalization, symbolization, and formalization of this arithmetic, thus this arithmetic provides a foundation for algebra;
- access to algebra can be democratized, because algebra understanding is not then limited to the few who have built a foundation for it without the help of the official curriculum.

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